3-qubit phase flip code

phase flip operator Z acts as

$$a|0\rangle + b|1\rangle \longmapsto a|0\rangle - b|1\rangle$$

with probability $p > 0$
(state is unchanged with prob. $1-p$)

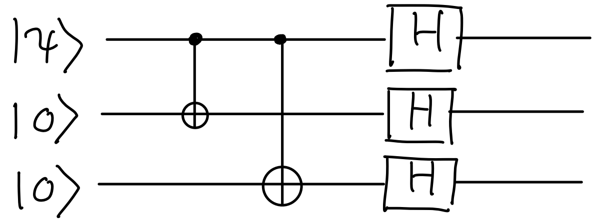
$\rightarrow$ work in qubit basis

$$|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}\ , \quad |-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$$

$\rightarrow$ $Z : |+\rangle \mapsto |-\rangle$

$\rightarrow$ use $|0_L\rangle \equiv |{+}{+}{+}\rangle$ and $|1_L\rangle \equiv |{-}{-}{-}\rangle$
as logical zero and one

$\rightarrow$ perform all operations of error-correction just as for bit flip channel
apply Hadamard gate to switch between $|+\rangle, |-\rangle$ and $|0\rangle, |1\rangle$ basi



Use projectors $P_j' \equiv H^{\otimes 3} P_j H^{\otimes 3}$ for error detection

The Shor code

How to protect against "arbitrary" error?

→ shor code

steps:

1) encode qubit using phase flip code

$|0\rangle \to |+++\rangle, \quad |1\rangle \to |---\rangle$

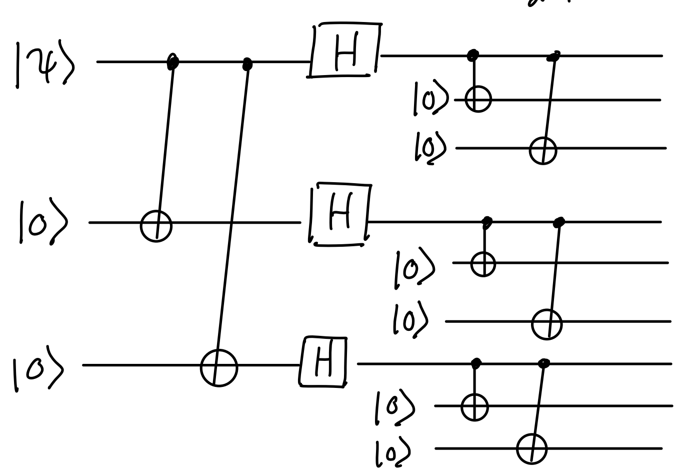
2) encode each of the 3 qubits using 3-qubit bit flip code

$$|+ \quad + \quad +\rangle \qquad\qquad |- \quad - \quad -\rangle$$

$$\frac{|000\rangle+|111\rangle}{\sqrt{2}} \quad \frac{|000\rangle+|111\rangle}{\sqrt{2}} \quad \frac{|000\rangle+|111\rangle}{\sqrt{2}} \qquad\qquad \frac{|000\rangle-|111\rangle}{\sqrt{2}}$$

$$|0\rangle \to |0_L\rangle \equiv \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \to |1_L\rangle \equiv \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$



concatenation circuit

Shor code is able to protect against phase flip and bit flip errors :

1) suppose a bit flip occurs on first qubit
   → perform $Z_1 Z_2$ measurement
   → produces sign change
     so either q_1 or q_2 has been flipped
   → measure $Z_2 Z_3$
   → no sign change
     so q_1 was flipped
   → recover from error by flipping first qubit again

2) Suppose a phase flip occurs on first qubit
   effect: $|000\rangle + |111\rangle \mapsto |000\rangle - |111\rangle$
   on first block
   → measure sign in first two blocks and last two blocks
   → overall sign flip in first case, no sign flip in second
   → restore by flipping sign in the first block of 3 qubits

3) Suppose both bit and phase flip error occurs on first qubit

→ $Z_1 X_1$ was applied

→ perform bit-flip and phase flip error correction successively

4) Protection against arbitrary errors

$$|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \longrightarrow \mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger$$

focus on single term in this sum

then $\quad E_i = e_{i0} I + e_{i1} X_1 + e_{i2} Z_1 + e_{i3} X_1 Z_1$

→ $E_i|\psi\rangle$ is superposition of 4 terms:
$|\psi\rangle$, $X_1|\psi\rangle$, $Z_1|\psi\rangle$, $X_1 Z_1|\psi\rangle$

→ measurement of error syndrome collapses this superposition into one of the 4 states

→ perform error correction as described above

→ recover $|\psi\rangle$

Stabilizer Codes

The code space of a stabilizer QEC code is defined by a stabilizer group $S = \langle \{S_i\} \rangle$.

→ mutually independent logical operators $\{L_j^Z\}$

→ computational basis state determined by $\langle \{S_i\}, \{(-1)^{m_j} L_j^Z\} \rangle$

→ find logical operators $\{L_j^X\}$ subject to

$$L_j^X L_i^Z = (-1)^{\delta_{ij}} L_i^Z L_j^X$$

→ $\{L_i^Z, L_i^X\}$ represents ith logical qubit

<u>example</u> : Shor code

$$\langle \{S_i\} \rangle = \begin{matrix} X & X & X & X & X & X & I & I & I \\ I & I & I & X & X & X & X & X & X \\ Z & Z & I & I & I & I & I & I & I \\ I & Z & Z & I & I & I & I & I & I \\ I & I & I & Z & Z & I & I & I & I \\ I & I & I & I & Z & Z & I & I & I \\ I & I & I & I & I & I & Z & Z & I \\ I & I & I & I & I & I & I & Z & Z \end{matrix}$$

$X_L = X^{\otimes 9}$ $\qquad Z_L = Z^{\otimes 9}$

For n qubits and $(n-k)$ stabilizers, we have k pairs of logical operators
$\rightarrow [n,k]$ stabilizer code

Definition:

1) The "Pauli group" G_n consists of <u>all</u> n-fold tensor products of Pauli matrices with multiplicative factors $\pm 1, \pm i$

$G_1 \equiv \{\pm I, \pm i I, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

2) Denote by $N(S)$ the normalizer of the stabilizer group S, i.e.

$$N(S) \equiv \{ E \in G_n \mid E g E^{\dagger} \in S \ \forall g \in S \}$$

We have $S \subseteq N(S)$

Theorem (Error-correction conditions for stabilizer codes):

Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^{\dagger} E_k \notin N(S) - S$ for all j and k. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.

Proof:

Let P be the projector onto the code space $C(S)$. For given j and k there are two possibilities: either $E_j^\dagger E_k$ in S or $E_j^\dagger E_k$ in $G_n - N(S)$. Consider the first case. Then $P E_j^\dagger E_k P = P$.

Suppose $E_j^\dagger E_k \in G_n - N(S)$

$\longrightarrow$ $E_j^\dagger E_k$ anticommutes with some element $g_1 \in S$

Let $g_1, \dots, g_{n-k}$ be a set of generators of S s.th.
$$P = \frac{\prod_{l=1}^{n-k} (I + g_l)}{2^{n-k}}$$

$$\Rightarrow E_j^\dagger E_k P = (I - g_1) E_j^\dagger E_k \prod_{n=2}^{n-k} \frac{(I + g_l)}{2^{n-k}}$$

But $P(I - g_1) = 0$ since $(I + g_1)(I - g_1) = 0$

$$\Rightarrow P E_j^\dagger E_k P = 0$$

Interpretation:

Suppose $g_1, \dots, g_{n-k}$ is a set of generators for the stabilizer of an $[n, k]$ stabilizer code.

Moreover, let $\{E_j\}$ be a correctable set of errors for the code.

→ error detection is performed by measuring g_1 through g_{n-k}

→ obtain error syndrom $\beta_1, \dots, \beta_{n-k}$ (measurement results)

If error E_j occured then

$$E_j^\dagger g_e E_j = \beta_e g_e$$

→ if E_j is unique error with this syndrome, then simply apply $E_j^\dagger$ for recovery

→ if $E_{j'}$ gives same syndrome ($j' \neq j$), then $E_j P E_j^\dagger = E_{j'} P E_{j'}^\dagger$

→ $E_j^\dagger E_{j'} P E_{j'}^\dagger E_j = P \Rightarrow E_j^\dagger E_{j'} \in S$

→ apply $E_j^\dagger$ after error $E_{j'}$ has occured to achieve recovery

(Note: E_j is picked arbitrarily from all errors with same syndrome!)

$\square$